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The Einstein-Dirac Equation in Robertson-Walker Space-Time Does Not Admit Standard Solutions

Antonio Zecca

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Abstract The Einstein-Dirac equation is considered in the Robertson-Walker space-time. Solutions of the equation are looked for in the class of standard solutions of the Dirac equation. It is shown that the Einstein-Dirac equation does not have standard solutions for both massive and massless Dirac field. Also superpositions of massive standard solutions are not solutions of the Einstein-Dirac equation. The result, that is briefly commented, is coherent and complementary to other existing results.

Keywords Dirac equation · Einstein-Dirac equation · Robertson-Walker space-time · Solutions

1 Introduction

One of the problem in general relativity is the study of the interaction of gravitation with fields endowed with spin. The formulation of the problem can be done in curved space-time by considering the equation of the field coupled to the Einstein field equation having the energy momentum tensor of the field as a source. The simplest situation corresponding to non zero spin is the Einstein-Dirac (E-D) equation. In connection to neutrino-gravitation interaction it has been noted that ghost neutrinos (namely solutions of the E-D equation that yield a zero momentum energy tensor [2]) do exist. These neutrino field solutions are possible in Petrov type D or N space-times [4]. Ghost neutrinos were introduced in cylindrically-symmetric space-time [3]. In that space-time other solutions have been found [10] that in particular contains the ghost neutrinos of Ref. [3]. On general ground, the E-D equation

A. Zecca (✉)
Dipartimento di Fisica dell' Universita', Via Celoria 16, 20133, Milano, Italy
e-mail: zecca@mi.infn.it

A. Zecca
INFN, Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy

A. Zecca
GNFM, Gruppo Nazionale per la Fisica Matematica, Sesto Fiorentino, Italy

has been studied in space-time for which the Dirac equation a priori admits separation of variables. In particular solutions have been given for Riemannian space of Sẗackel type [1].

The E-D equation has been also studied specifically in the Robertson-Walker (R-W) space-time. There no massless solutions have been found in the class of the solutions of the Dirac equation proportional to a constant 4-spinor and only ghost massive solutions have been found [5]. There are however in the literature solutions of the Dirac equation in the two-spinor form, not equivalent to those just mentioned (e.g., [6, 13, 14]).

The object of the present paper is to look for solutions of the E-D equation in R-W space-time in the class of the solutions of the Dirac equation determined in [15] and called hereafter standard solutions. The standard solution are variable separated and regular in their angular dependence. Their properties together with the special diagonal form of the Einstein tensor in R-W metric, imply that the E-D equation cannot have standard solutions both in case of massive and massless fermion field. For massive fermions it is shown that also superposition of standard solution cannot be solutions of the E-D equation. It is left open the problem whether superposition of massless standard solutions can be solutions of the E-D equation.

2 Formulation of the Problem and Preliminary Results

The solution of the Einstein-Dirac equation in Robertson-Walker space-time of metric [12]

$$ds^2 = dt^2 - R(t)^2\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right], \quad a = 0, \pm 1 \tag{1}$$

consists in simultaneously solving the Einstein and Dirac field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\alpha_\alpha = 8\pi G T_{\mu\nu}(\varphi, \chi), \tag{2}$$

$$\nabla^D_{\dot{X}}\varphi_D + \mu_\star\chi_{\varphi_A} = 0, \qquad \nabla^{\dot{Z}}_A\chi_{\dot{Z}} - \mu_\star\varphi_D = 0 \tag{3}$$

$\mu_\star = im_0/\sqrt{2}, m_0$ being the mass of the fermions and $T_{\mu\nu}(\varphi,\chi)$ the energy momentum tensor of the Dirac field [7, 11]:

$$T_{\mu\nu} = d\big\{-\bar{\varphi}_{\dot{X}}\sigma^{A\dot{X}}_{(\mu}\nabla_{\nu)}\varphi_A + \varphi_A\sigma^{A\dot{X}}_{(\mu}\nabla_{\nu)}\bar{\varphi}_{\dot{X}} + \chi_{\dot{X}}\sigma^{A\dot{X}}_{(\mu}\nabla_{\nu)}\bar{\chi}_A - \bar{\chi}_A\sigma^{A\dot{X}}_{(\mu}\nabla_{\nu)}\chi_{\bar{X}}\big\} \tag{4}$$

$$\begin{aligned} &= \frac{d}{2}\sigma^{A\dot{X}}_\mu\sigma^{B\dot{Y}}_\nu\big\{-\bar{\varphi}_{\dot{X}}\nabla_{B\dot{Y}}\varphi_A - \bar{\varphi}_{\dot{Y}}\nabla_{A\dot{X}}\varphi_B + \varphi_A\nabla_{B\dot{Y}}\bar{\varphi}_{\dot{X}} + \varphi_B\nabla_{A\dot{X}}\bar{\varphi}_{\dot{Y}} \\ &\quad + \chi_{\dot{X}}\nabla_{B\dot{Y}}\bar{\chi}_A + \chi_{\dot{Y}}\nabla_{A\dot{X}}\bar{\chi}_B - \bar{\chi}_A\nabla_{B\dot{Y}}\chi_{\dot{X}} - \bar{\chi}_B\nabla_{A\dot{X}}\chi_{\dot{Y}}\big\} \end{aligned} \tag{5}$$

$$= \frac{d}{2}\sigma^{A\dot{X}}_\mu\sigma^{B\dot{Y}}_\nu T_{A\dot{X}B\dot{Y}} \tag{6}$$

where d is a parameter and the $\sigma^{A\dot{X}}_\mu$'s are the Infeld-van der Waerden symbols [11]. The trace $T = T^\mu_\mu(\varphi,\chi)$ can be calculated by using the fields equation so obtaining

$$T = -2d\mu_\star\big(\chi_{\dot{X}}\bar{\varphi}^{\dot{X}} - \varphi_A\bar{\chi}^A\big). \tag{7}$$

The Dirac equation (3) can be separated by setting (e.g., [13, 15])

$$(\varphi_1, -\varphi_0) = \frac{\exp(im\varphi)}{rR}\big(H_1(r,t)S_1(\theta), H_2(r,t)S_2(\theta)\big),$$
$$(\chi_0, \chi_1) = \frac{\exp(im\varphi)}{rR}\big(H_2(r,t)S_1(\theta), H_1(r,t)S_2(\theta)\big) \tag{8}$$

where $m = 0, \pm 1, \pm 2, \dots$. The angular functions satisfy: $L^- S_2 = -\lambda S_1$, $L^- S_1 = \lambda S_2$, $(L^\pm = \partial_\theta \mp \csc\theta + \frac{1}{2}\cot\theta)$. The regular solutions have the form $S_i = S_{ilm}(\theta)$ $(i = 1, 2)$, $l = |m|, |m|+1, |m|+2, \dots$. For $|m| \le 1$, $\lambda^2 = (l + 1/2)^2$ while for $m = 0$, $\lambda^2 = (l+1)^2$. Moreover $S_i = \bar{S}_i$, $S_{2lm} = S_{1l-m}$. The functions $S^{(j)}_{lm} = \exp(im\varphi)S_{jlm}$ are a complete orthonormal set in $L^2(\Omega) : (S^{(j)}_{lm}, S^{(j)}_{l'm'}) = \delta_{ll'}\delta_{mm'}$ $(j = 1, 2)$ [15]. The r, t dependence is further separated by setting $H_1 = F(r)T(t) + G(r)S(t)$, $H_2 = F(r)T(t) - G(r)S(t)$. The corresponding radial equations are

$$\sqrt{1-ar^2}G' - \frac{\lambda}{r}G = -kF,$$
$$\sqrt{1-ar^2}F' + \frac{\lambda}{r}F = -kG \tag{9}$$

k the separation constant. The radial equation can be integrated exactly [15]. The solutions have the property $G = G_k(r,\lambda) \cong F_k(r,-\lambda)$ and $G(r) \to r^\lambda$, $F(r) \to r^{-\lambda}$ for $r \to 0$. Finally the time equations can be disintangled. The solutions have the property $T_k(t) \cong \bar{S}_k(t)$. For $m_0 = 0$, $S \cong T \cong \bar{T}$. The solutions of the Dirac equation determined in this way will be reported as standard solutions. Note that non standard solutions exist. Indeed for $\lambda = m_0 = k = 0$ one has

$$S_1 \cong (\sin\theta)^{1/2}|\tan\theta/2|^m, \qquad S_2 \cong (\sin\theta)^{-1/2}|\tan\theta/2|^m,$$
$$G = G_0 = const., \qquad F = F_0 = const., \tag{10}$$
$$T(t) \cong S(t) = \frac{A}{\sqrt{R(t)}}, \qquad A = const.$$

The object is now to establish whether the E-D equation admits of solution in the class of the standard solutions of the Dirac equation.

3 The E-D Equation Has No Standard Solutions

In case of the R-W metric the Ricci scalar does not depend on the spatial variables and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\delta_\delta$ results to be diagonal. From these facts and by taking trace into (2), the conditions

$$\partial_j T^\alpha_\alpha = 0, \qquad j = \varphi, \theta, r, \tag{11}$$

$$T_{\mu\nu} = 0, \qquad \mu \ne \nu \tag{12}$$

must be therefore fulfilled by the energy momentum tensor.

In case of massive Dirac field, the expression (7) becomes, on the standard solutions,

$$T^\alpha_\alpha = -\frac{2d\mu_\star}{r^2R^2}(\bar{H}_1H_2 + \bar{H}_2H_1)(S_1\bar{S}_1 + S_2\bar{S}_2) \tag{13}$$

$$= -4d\mu_\star\frac{T\bar{T}}{r^2R^2}(S_1\bar{S}_1 + S_2\bar{S}_2)[|F(r,\lambda)|^2 - |F(r,-\lambda)|^2]. \tag{14}$$

Therefore, if $m_0 \neq 0$, the condition (11) cannot be verified on account of the explicit expression of the angular functions (see e.g. [8]) and the fact that $|F(r,\lambda)| \neq |F(r,-\lambda)|$ (see [15]).

If, conversely, $m_0 = 0$, the condition (11) is automatically satisfied and one has to check whether the conditions (12) are verified. To that end, consider the null tetrad frame under which the standard solutions have been obtained [15]. In that tetrad the σ-matrices result proportional the Pauli matrices plus the identity matrix $\mathbf{1}_{2\times 2}$ (the explicit form of the proportionality coefficients is not relevant here). From equations (6), (12) there follows that $T_{A\dot{X}B\dot{Y}}$ must vanish for some values of the indexes. In particular one can see that

$$T_{0\mathrm{i}0\dot{0}} = 0. \tag{15}$$

To explicit this condition on the standard solutions, we consider that

$$\begin{aligned}
&\nabla_{0\dot{0}}\varphi_0 = (\partial_{0\dot{0}} - \epsilon)\varphi_0, && \nabla_{0\mathrm{i}}\varphi_0 = (\partial_{0\mathrm{i}} - \beta)\varphi_0,\\
&\nabla_{0\dot{0}}\bar{\varphi}_{\mathrm{i}} = (\partial_{0\dot{0}} + \epsilon)\bar{\varphi}_{\mathrm{i}}, && \nabla_{0\mathrm{i}}\bar{\varphi}_{\dot{0}} = (\partial_{0\mathrm{i}} + \beta)\bar{\varphi}_{\dot{0}} + \rho\bar{\varphi}_{\mathrm{i}},\\
&\partial_{0\dot{0}} = \frac{1}{\sqrt{2}}\left(\partial_t + \frac{\sqrt{1-ar^2}}{R}\partial_r\right), && \partial_{0\mathrm{i}} = \frac{1}{rR\sqrt{2}}(\partial_\theta + i\csc\theta\,\partial_\varphi),\\
&\rho = -\frac{1}{\sqrt{2}}\left(\frac{\dot{R}}{R}\frac{\sqrt{1-ar^2}}{rR}\right), && \epsilon = \frac{1}{\sqrt{2}}\frac{\dot{R}}{R},
\end{aligned} \tag{16}$$

here $\partial_{A\dot{A}}$ and ϵ, β are, respectively, the directional derivatives and spin coefficients in the mentioned tetrad [8]. By direct calculations there follows

$$\begin{aligned}
T_{0\mathrm{i}0\dot{0}} &= \varphi_0\partial_{0\dot{0}}\bar{\varphi}_{\mathrm{i}} - \bar{\varphi}_{\mathrm{i}}\partial_{0\dot{0}}\varphi_0 + \varphi_0\partial_{0\mathrm{i}}\bar{\varphi}_{\dot{0}} - \bar{\varphi}_{\dot{0}}\partial_{0\mathrm{i}}\varphi_0 + 2\epsilon\varphi_0\bar{\varphi}_{\mathrm{i}} + 2\beta\varphi_0\bar{\varphi}_{\dot{0}} + \rho\varphi_0\bar{\varphi}_{\mathrm{i}}\\
&\quad + \chi_{\mathrm{i}}\partial_{0\dot{0}}\bar{\chi}_0 - \bar{\chi}_0\partial_{0\dot{0}}\chi_{\mathrm{i}} + \chi_{\dot{0}}\partial_{0\mathrm{i}}\bar{\chi}_0 - \bar{\chi}_0\partial_{0\mathrm{i}}\chi_{\dot{0}} - 2\epsilon\chi_{\mathrm{i}}\bar{\chi}_0 - 2\beta\chi_{\dot{0}}\bar{\chi}_0 - \rho\bar{\chi}_0\chi_{\mathrm{i}}\\
&= \frac{\sqrt{1-ar^2}}{\sqrt{2}(rR)^3} S_1 S_2(\bar{H}_2 H_1 + H_2\bar{H}_1)\\
&\quad + \frac{\bar{H}_2 H_2}{\sqrt{2}(rR)^3}\left[\frac{\cos\theta}{\sin\theta}(S_2\bar{S}_2 - S_1\bar{S}_1) + \frac{2m}{\sin\theta}(S_2\bar{S}_2 + S_1\bar{S}_1)\right]\\
&\quad + 2\frac{\sqrt{1-ar^2}}{\sqrt{2}rR^3} S_2\bar{S}_1\left[-H_2\partial_r\left(\frac{\bar{H}_1}{r}\right) + \bar{H}_1\partial_r\left(\frac{H_2}{r}\right)\right.\\
&\quad \left. + H_1\partial_r\left(\frac{\bar{H}_2}{r}\right) - \bar{H}_2\partial_r\left(\frac{H_1}{r}\right)\right].
\end{aligned} \tag{17}$$

Suppose now $m \neq 0$. By variable separation the condition (15) applied to (17) gives

$$\cos\theta(S_1^2 - S_2^2) + 2m(S_1^2 + S_2^2) = C\sin\theta S_1 S_2 \tag{18}$$

(C a constant) a constraint that cannot be satisfied by $S_1,\ S_2$. Indeed a series expansion for $\theta \to 0$ of the angular functions (see [8]) in (18) implies $m = 1/2$, a value that is not possible for the standard solutions.

Let now $m = 0$, then $S_1 \cong S_2$ and $|S_1|^2 = |S_2|^2$ from the angular equation (see Sect. 1). The condition (15) gives now

$$\frac{H_2\bar{H}_1 + H_1\bar{H}_2}{r^2} = H_2\partial_r\left(\frac{\bar{H}_1}{r}\right) - \bar{H}_1\partial_r\left(\frac{H_2}{r}\right) - H_1\partial_r\left(\frac{\bar{H}_2}{r}\right) + \bar{H}_2\partial_r\left(\frac{H_1}{r}\right). \tag{19}$$

By further using $H_1 = FT + GS$, $H_2 = FT - GS$ and (9), the expression (19) reduces to

$$\sqrt{1-ar^2}\frac{F\bar{F}-G\bar{G}}{r} = \pm 2\left[k(G\bar{G}-F\bar{F}) + \frac{\lambda}{r}(F\bar{G}+G\bar{F})\right]. \tag{20}$$

The double sign refers, respectively, to $T = \pm S$ a relation that follows from the time equations [15]. The invariance of (19) under the substitution $(H_1, H_2) \to (H_1, -H_2)$, implies invariance of (20) under $F \leftrightarrow G$. But this gives $|F| = |G|$ or $\sqrt{1-ar^2} = 2kr$, that are both impossible conditions.

4 Generalizations

We now show that also solutions of the E-D equation that are superpositions of massive standard solutions of the Dirac equation are not possible. Suppose indeed φ_A, $\chi_{\dot{B}}$ be solutions of the E-D equation such that

$$\begin{aligned} \varphi_A &= \sum_{lm}\int dk\, c_{lm}(k)\varphi_{Alm}, \\ \chi_{\dot{B}} &= \sum_{lm}\int dk\, c_{lm}(k)\chi_{Alm} \end{aligned} \tag{21}$$

with $\int dk c_{lm}(k)T_k(t) \neq 0$ for some t, l, m. Under these assumptions, the expression (7) becomes

$$\begin{aligned} T^\alpha_\alpha = -2d\mu_\star \sum_{lml'm'}\Bigg\{ \frac{e^{i(m-m')\varphi}}{r^2R^2}\Bigg[S_{1lm}S_{1l'm'}\int dk H_{2kl}c_{lm}(k) \\ \times \int dk'\bar{H}_{1k'l'}\bar{c}_{l'm'}(k') + (1\leftrightarrow 2\Bigg] + C.C.\Bigg\}. \end{aligned} \tag{22}$$

Since T^α_α cannot depend on r, one has $\int d\Omega \partial_r T^\alpha_\alpha = 0$. By using the orthogonality relations of the angular function $S^{(i)}_{lm}$, the decomposition $H_1 = FT + GS$, $H_2 = FT - GS$ and the behaviour of F, G for $r \to 0$ into (22), one is left with

$$r^{-2\lambda-3}\sum_{lm}(2\lambda+2)\left|\int dk T_{kl}c_{lm}(k)\right|^2 + r^{2\lambda-2}\sum_{lm}(2\lambda-2)\left|\int dk S_{kl}c_{lm}(k)\right|^2 \cong 0 \tag{23}$$

a condition that cannot be satisfied for $r \to 0$ because it implies $\int dk T_k c_{lm}(k) = 0\ \forall t, l, m$.

5 Remarks and Comments

We have shown that no massless and no massive standard solutions and no superpositions of massive standard solutions exist of the Einstein-Dirac equation in R-W metric. The result is however coherent with results of previous papers [5, 14] where solutions have been explicitly given. The example of [14] is in fact a non standard solution. In Ref. [5] it was shown that no solution of the E-D equation exist for massless fermions and only null ghost solutions exist for the massive fermions. This does not contradict our result because all those solutions are proportional to a constant 4-spinor, a situation that cannot happen in the present study.

Indeed this would imply, e.g., $H_1 S_1 \propto H_2 S_2$, a condition that is not satisfied by the standard solutions. The present scheme is therefore in some sense complementary to the mentioned ones. It would be however interesting to establish whether there are solutions of the E-D equation that are superposition of massless standard solutions, a problem that is left open here. Also the existence of solutions that are not standard nor of the form of those found in [5] would be interesting. In looking for those possibilities one should not renounce to the orthogonality of the solutions. Indeed this is an essential feature of the standard solutions that enables to define the normal modes in order to quantize the Dirac field [9].

For what concerns the physical interpretation of the results, we note the negative impact it has in cosmology. The presence of spin $1/2$ field alone is not sufficient to produce the gravitation of an isotropic homogeneous cosmological model (as anyone based on the R-W metric [12]).

The general treatment of the interaction of spin $1/2$ field and gravity is very wide and goes beyond the E-D equation scheme. We only note that a general study of massive neutrinos propagation and interference in Schwarzschild space-time can be found in [17] (see also references therein). It is also possible to formulate the problem in space-time with torsion. In case of Weitzenböck space-time the interaction can be incorporated in the spin connection. This leads to results for mass neutrino flavour evolution equation that are equivalent to those of general relativity in case of spherical symmetry but are different for non zero axial symmetry (see [16] and references therein).

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